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On the Geometric Properties of AdS Instantons

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Abstract

According to the positive energy conjecture of Horowitz and Myers, there is a specific supergravity solution, AdS soliton, which has minimum energy among all asymptotically locally AdS solutions with the same boundary conditions. Related to the issue of semiclassical stability of AdS soliton in the context of pure gravity with a negative cosmological constant, physical boundary conditions are determined for an instanton solution which would be responsible for vacuum decay by barrier penetration. Certain geometric properties of instantons are studied, using Hermitian differential operators. On a d-dimensional instanton, it is shown that there are $d-2$ harmonic functions. A class of instanton solutions, obeying more restrictive boundary conditions, is proved to have $d-1$ Killing vectors which also commute. All but one of the Killing vectors are duals of harmonic one-forms, which are gradients of harmonic functions, and do not have any fixed points.

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1 Introduction

Like for any other field theory, positivity of the energy is a necessary condition for ensuring the stability of general relativity. In the presence of gravity, although the notion of local energy density is not well defined, one can still talk about the total energy of a gravitating system defined in terms of the asymptotic behavior of the metric with respect to a background geometry [1],[2]. For asymptotically flat spaces, under general assumptions, the complete proof of positive energy theorem was first given in [3]. Later, a simple and elegant proof was presented in [4], using spinors. On the other hand, it is known that there are non-trivial zero [5] and negative energy [6] asymptotically flat spaces on which these proofs do not apply. Generically, such a solution has an asymptotic circle S^1 which is contractible in the interior, thus the asymptotic geometry is only locally Minkowskian. The proof of [3] does not apply due to this topological difference, and spinors used in the proof given in [4] are not well defined on such spaces [5]. This sector of the theory is completely unstable.

Following the remarkable conjecture of [7],[8],[9], there is now considerable evidence for a correspondence between certain string theories on Anti-de-Sitter (AdS) spaces and conformal field theories (CFT) living on the boundaries of these spaces. For some applications, the correspondence allows us to work with supergravity approximations to string theories. Although supersymmetry plays a crucial role for the conjecture, in [10], it has been suggested that non-supersymmetric Yang-Mills gauge theory can also be described by AdS/CFT correspondence by compactifying one direction on S^1 and imposing supersymmetry breaking boundary conditions to fermions. Recently, Horowitz and Myers have studied the consequences of this proposal from the point of view of supergravity theory on AdS [11]. By AdS/CFT correspondence, one should consider geometries which have one spatial direction compactified on S^1 asymptotically. Thus, the asymptotic geometry is not globally but only locally the AdS space. The standard techniques can be applied to show the positivity of energy for spaces which approach globally to AdS at infinity [12]. However, the stability of gravity is now questionable with this new asymptotic form, like in the asymptotically locally Minkowskian sector. Indeed, Horowitz and Myers presented a solution which is completely regular and has negative total energy. Following Horowitz and Myers, we will also refer to this solution as AdS *soliton*. Not surprisingly, the topology of AdS soliton is such that the asymptotic circle at infinity is contractible in the interior. Horowitz and Myers have beautifully identified the negative energy of this solution with the negative Casimir energy of the field theory which arises due to the breaking of supersymmetry. Then, the expected stability of non-supersymmetric Yang-Mills gauge theory led them to conjecture a new positive energy theorem for asymptotically locally AdS spaces.

Perturbative stability of AdS soliton (up to quadratic metric fluctuations) has been shown in [11]. A related and important problem is the issue of semiclassical stability. In field theory, a false vacuum decay by barrier penetration to a stable ground state via instanton like solutions of Euclidean field equations, in which at large distances the fields take their values in the false vacuum [13]. Therefore, one should study the possible Euclidean metric solutions of Einstein equations which approach the Euclidean AdS soliton (which will also be referred as AdS soliton) asymptotically. The existence of such a solution (which will be referred simply as an instanton) may then imply the instability of the AdS soliton as a result of a semiclassical analysis (i.e. studying the small fluctuations around the instanton). With the above motivation, in this paper, we will try to determine some geometric properties of instanton solutions by studying certain differential operators. Similar arguments were first used to show the semiclassical stability of Minkowski space in [4].

The organization of the paper is as follows. In section 2, we discuss the definition of the Euclidean action, which can be expressed as a surface integral when being evaluated on a background, and fix the asymptotic behavior of the metric. In section 3, we introduce two Hermitian differential operators defined on divergence free vector fields, and determine their zero modes. One of the operators has no zero modes and turns out to be invertible. We also study the properties of the Laplacian acting on square integrable functions. In section 4, by solving Dirichlet problems related to the invertible differential operators introduced in section 3, we try to prove that an instanton has certain number of harmonic functions and commuting Killing vectors. We solve the Dirichlet problem related to the Laplacian and show that on a d-dimensional instanton there are $d-2$ harmonic functions. The Dirichlet problem, which is intimately related to the existence of Killing vectors, can be solved in six or higher dimensional instantons. However, that solution does not have appropriate asymptotics to give the desired Killing vectors. Then, we focus on a family of instantons obeying more restrictive boundary conditions. On a d-dimensional instanton of this family, we show that there are $d-1$ commuting Killing vectors. It turns out that all but one of the Killing vectors are duals of harmonic one-forms, which are gradients of harmonic functions, and do not have any fixed points. We conclude with some brief remarks in section 5.

2 Euclidean action, surface terms and boundary conditions

The Lorentzian action of the pure gravity on a d-dimensional spacetime M_L with a negative cosmological constant, $\Lambda < 0$, may be written as:

$$I_L = \frac{1}{16\pi G}\int_{M_L}(R-2\Lambda) + \frac{1}{8\pi G}\int_{\partial M_L} K, \tag{1}$$

where G is the gravitational constant, R is the scalar curvature of the metric g_{AB}, K is the trace of the extrinsic curvature of the boundary ∂M_L. The volume elements on M_L and ∂M_L are determined by g_{AB} and the induced metric on ∂M_L, respectively, and will not be written explicitly. The surface term is added to obtain the correct equations subject only to the condition that the induced metric on ∂M_L is held fixed.

Inspired by the Lorentzian form of the action, one may define the Euclidean action on a manifold M with boundary ∂M as:

$$I_E = -\frac{1}{16\pi G}\int_{M}(R-2\Lambda) - \frac{1}{8\pi G}\int_{\partial M} K, \tag{2}$$

which, upon variation, gives the vacuum Einstein equations with (++...+) signature:

$$R_{AB} - \frac{1}{2}g_{AB}R + \Lambda g_{AB} = 0. \tag{3}$$

In Euclidean quantum gravity, the central object is the partition function, Z, defined by the path integral over all metrics:

$$Z = \int D[g]e^{-I_E[g]}. \tag{4}$$

In a semiclassical approximation, the dominant contribution to Z will come from fluctuations around a saddle point of the action. However, the Euclidean action (2) diverges for non-compact geometries, since the Einstein equations imply that R is constant so the integral is just a multiple of the infinite volume of the space. Furthermore, one should also impose some boundary conditions to fluctuations around such spaces to define the path integral properly. To have a well defined Z then, one can pick up a background geometry $\bar{g}_{AB}$, obeying (3), and use a modified action

$$\tilde{I}_E[g] = I_E[g] - I_E[\bar{g}] \tag{5}$$

in the definition of the partition function:

$$\tilde{Z} = \int D[g] e^{-\tilde{I}_E[g]}. \tag{6}$$

This new partition function $\tilde{Z}$ is related to Z by a normalization with an infinite constant. If one includes in the path integral the geometries which approach the background geometry at infinity such that $\tilde{I}_E[g]$ is finite, then $\tilde{Z}$ becomes well defined. For a saddle point, using Einstein equations and up to a finite constant [2], $\tilde{I}_E$ may be rewritten as:

$$\tilde{I}_E = -\frac{1}{8\pi G} \int_{\partial M} (K - \bar{K}), \tag{7}$$

where $\bar{K}$ is the trace of the extrinsic curvature of the boundary embedded in the background space.

In our case the background space is the d-dimensional AdS soliton which is given by [11](in Euclidean signature):

$$ds^2 = a^2 r^2 \left[\left(1 - \frac{r_0^{p+1}}{r^{p+1}} \right) d\tau^2 + (dx^i)^2 \right] + \frac{1}{a^2 r^2} \left(1 - \frac{r_0^{p+1}}{r^{p+1}} \right)^{-1} dr^2, \tag{8}$$

where r is a radial coordinate restricted to $r > r_0$, τ is an angular coordinate, x^i (with $i = 1...p$ and $d = p + 2$) are coordinates on p-dimensional Euclidean space R^p and a is the inverse radius which is related to cosmological constant by $a^2 = -2\Lambda/(d-1)(d-2)$. To avoid a conical singularity at $r = r_0$ one should identify τ with period $\beta = 4\pi/(a^2(p+1)r_0)$. We also define $l^2 = a^2(d-1)$ so that the Einstein equations (3) may be written as:

$$R_{AB} = -l^2 g_{AB}. \tag{9}$$

The $p = 3$ solution is related to the Yang-Mills gauge theory in 4-dimensions with one compactified direction. On the other hand, the $p = 5$ case is related to the exotic 6-dimensional world-volume theory of coincident M-theory fivebranes with one compact direction, which reduces to Yang-Mills gauge theory in four-dimensions by a further compactification.

In using this form of the metric in a semiclassical analysis, one faces two important technical problems. The first one is that the boundary at $r \to \infty$ of AdS soliton is

[2] Like in [14], (5) and (7) may not be equal when the integration regions of the background and the saddle point are different. In this case an additional but finite contribution should be added to (7). However, this point will not alter our discussion of boundary conditions for which we will mainly use (7).

$R^p \times S^1$. Therefore, the modified action (7) may also diverge even for suitably well defined boundary conditions due to the integral over R^p. The second closely related problem is that the interior of AdS soliton (regions of finite r) is also non-compact which implies the existence of other boundaries in addition to boundary at $r \to \infty$ (like the boundary at $x^i x^i \to \infty$). One should also impose certain boundary conditions along these hypersurfaces.

There are different ways of overcoming these technical problems. One may further redefine the partition function (6) by replacing $\tilde{I}_E$ by $\tilde{I}_E/\mathcal{V}$, where $\mathcal{V}$ is the volume of the space spanned by the coordinates x^i. Then, one should impose the boundary conditions that $\partial_i g_{AB} = 0$ when $x^i x^i$ becomes greater then a constant. This ensures that $\tilde{I}_E/\mathcal{V}$ does not diverge, which in turn gives a well defined partition function.

Instead of dealing with action densities, we will proceed by modifying the AdS soliton by identifying the coordinates x^i with period L. This modification alters the asymptotic boundary from $R^p \times S^1$ to $T^p \times S^1$. One should not then worry about the two problems mentioned above: the asymptotic integrals are now over $T^p \times S^1$ which will be convergent for suitable boundary conditions at $r \to \infty$, and the interior region becomes compact; the only boundary is now located at $r \to \infty$. The original AdS soliton may be recovered by letting L go to infinity.

Asymptotic boundary conditions will be fixed by demanding that (7) be finite for an instanton. Before discussing these boundary conditions, let us point out another physical motivation for this choice. There is a close connection between (7) and the total energy defined in the Lorentzian context. Suppose we find an instanton solution, which asymptotically becomes the AdS soliton, and try to determine to which Lorentzian spacetime the AdS soliton decays (if it does). To find this spacetime, one should make an inverse analytical continuation of the instanton [5]. Assuming that $x^1 = 0$ is a plane of symmetry, this may be achieved by sending $x^1 \to ix^1$. The coordinate x^1 now plays the role of time and the energy of the Lorentzian solution, calculated with respect to the AdS soliton, turns out to be finite, when(7) being evaluated on the instanton is finite. This may easily be seen from the definition of total energy given in [15] and shows that the physically interesting instantons have *finite actions*. Note that a similar relation between the value of the Euclidean action and energy is also encountered in the asymptotically flat context.

Let us now turn to the determination of the asymptotic behavior of a possible instanton solution. It is assumed that outside a compact region, one can introduce angular coordinates $x^\alpha \equiv \tau, x^i$ (with periods β and L respectively) and r in which the metric becomes asymptotically the AdS soliton:

$$g_{AB} = \bar{g}_{AB} + h_{AB}, \quad h_{AB} \to 0 \quad as \;\; r \to \infty, \tag{10}$$

such that (7) is finite. We will make no assumptions about the topology of the compact region. In the asymptotic region and to the leading order in h_{AB}, the inverse of the metric can be found as:

$$g^{AB} = \bar{g}^{AB} - \bar{g}^{AC} h_{CD} \bar{g}^{DB}. \tag{11}$$

The volume form on the boundary grows like r^{p+1}, therefore, to have a well defined Euclidean action we should demand that at large r, $K - \bar{K} = O(1/r^{p+1})$. Since the boundary is a $r = constant$ hypersurface, the unit normal vector, n_A, to this surface is given by:

$$n_A = \frac{1}{\sqrt{g^{rr}}} \delta^r_A. \tag{12}$$

Using $K = \nabla_A n^A$ and further assuming that the r derivatives of the metric fall off one power faster, the finiteness of (7) implies for large r: [3]

$$h_{\alpha\beta} = O(\frac{1}{r^{p-1}}), \quad h_{\alpha r} = O(\frac{1}{r^p}), \quad h_{rr} = O(\frac{1}{r^{p+3}}). \tag{13}$$

One can also calculate the asymptotic form of the Christoffel symbol as:

$$\Gamma^r{}_{\alpha\beta} = -a^2 r^3 \delta_{\alpha\beta} + O(\frac{1}{r^{p-2}}), \quad \Gamma^r{}_{\alpha r} = O(\frac{1}{r^{p-1}}), \tag{14}$$

$$\Gamma^r{}_{rr} = -\frac{1}{r} + O(\frac{1}{r^{p+2}}), \quad \Gamma^\alpha{}_{\beta\delta} = O(\frac{1}{r^{p-1}}), \tag{15}$$

$$\Gamma^\alpha{}_{\beta r} = \frac{1}{r}\delta^\alpha_\beta + O(\frac{1}{r^{p+2}}), \quad \Gamma^\alpha{}_{rr} = O(\frac{1}{r^{p+3}}). \tag{16}$$

As it was pointed out in [16], the perturbation $h_{\alpha r}$ can be made to fall off faster than $1/r^p$ by an appropriate choice of coordinates at infinity.[4] Since this will not change the main conclusions of the paper, we do not fix this gauge freedom at this point. In the same paper, the physical boundary conditions were determined for a Lorentzian metric approaching asymptotically to AdS_4.

[3] If we did not periodically identify the coordinates x^i, then we had to impose the boundary conditions $\partial_i g_{AB} = 0$ when $x^i x^i$ is greater then a constant.

[4] I thank to M. Henneaux for pointing this out to me.

Compared to the asymptotically flat instanton like solutions [5], the metric approaches the background geometry more rapidly. We will use (13)-(16) at several different points. In most cases these will be sufficient for us to obtain the results we are seeking. On the other hand, at one important instance when we try to establish the existence of Killing vectors, we will be forced to consider more restrictive boundary conditions.

3 Hermitian differential operators and zero modes

We are searching properties of possible Euclidean metric solutions of (9) obeying the asymptotic boundary conditions determined in the previous section. Let us denote one of such solutions by M. In this section, we will introduce and study certain differential operators and mainly try to see if they are invertible on suitable Hilbert spaces.

Operators L_1 and L_2

We define the following differential operators on the tangent bundle of M:

$$L_1 k_A \equiv \nabla^2 k_A - l^2 k_A, \tag{17}$$

$$L_2 k_A \equiv \nabla^2 k_A + l^2 k_A, \tag{18}$$

which will act on the divergence free vector fields,

$$\nabla_A k^A = 0. \tag{19}$$

The reason for introducing these operators will become clear as we proceed. The vectors satisfying (19) is a subspace of the total space of vector fields and will be denoted by V. Let us first show that L_1 and L_2 are well defined operators on this subspace:

$$L_1 : V \to V, \tag{20}$$

$$L_2 : V \to V. \tag{21}$$

To verify this, we start from the curvature identity

$$(\nabla_A \nabla_B - \nabla_B \nabla_A)\nabla_C k_D = R_{ABC}{}^E \nabla_E k_D + R_{ABD}{}^E \nabla_C k_E, \tag{22}$$

and contract this with g^{BC} and g^{AD} to obtain

$$\nabla^A \nabla^2 k_A - \nabla^B \nabla^A \nabla_B k_A = 0. \tag{23}$$

Now, another curvature identity together with the Einstein equations (9) give:

$$\nabla^A \nabla_B k_A = \nabla_B \nabla^A k_A - l^2 k_B. \tag{24}$$

Using this in the second term of the (23), one obtains (for an arbitrary vector field k_A),

$$\nabla^A \nabla^2 k_A = \nabla^2 \nabla^A k_A - l^2 \nabla^B k_B. \tag{25}$$

If $k_A \in V$ then $\nabla^A k_A = 0$ and the above equation gives $\nabla^A \nabla^2 k_A = 0$. In this case, the definitions of L_1 and L_2 imply:

$$\nabla^A (L_1 k_A) = 0, \tag{26}$$

$$\nabla^A (L_2 k_A) = 0, \tag{27}$$

showing that $L_1 k_A \in V$ and $L_2 k_A \in V$ when $k_A \in V$. This verifies that L_1 and L_2 are well defined operators on V.

We introduce the following inner product on the vector space V:

$$< k | l >= \int_M g^{AB} k_A l_B. \tag{28}$$

In general, there are vectors in V which do not have well defined inner products. Thus, some boundary conditions should be imposed to eliminate these vector fields. The fact that, when r is large, the volume form on M grows like r^p and the asymptotic behavior of the metric fixed in the previous section suggest the following boundary conditions on the components of k_A (in x^α, r coordinates):

$$k_r = O(\frac{1}{r^{p/2+2}}), \qquad k_\alpha = O(\frac{1}{r^{p/2}}). \tag{29}$$

We $redefine$ V to be the space of divergence free vector fields obeying these boundary conditions. With this redefinition, the inner product (28) becomes well defined on V. Note that V is the kernel of the elliptic operator $*d*$ acting on the one-forms. Due to this fact, we assume that V satisfies the axioms of Hilbert spaces rigorously. We also assume that the r-derivatives of the components fall off one power faster.

Let us show that L_1 and L_2 are Hermitian operators on V. This requires vanishing of the following surface integral,

$$\int_{\partial M} n^A k^B (\nabla_A l_B) = 0, \tag{30}$$

for *all* vectors $k_A, l_A \in V$, where n^A is the unit normal to ∂M. The volume form on ∂M grows like r^{p+1} at large r. Using the expression (12) for the unit normal vector, the asymptotic behavior of the metric and the boundary conditions (29) obeyed by the vector fields k_A and l_A, one can check that the integrand vanishes as $r \to \infty$. Then, (30) implies, upon integration by parts,

$$\int_M k^A \nabla^2 l_A = \int_M l^A \nabla^2 k_A, \tag{31}$$

which in turn proves that L_1 and L_2 are Hermitian operators on V:

$$< k|L_1 l > = < L_1 k|l >, \tag{32}$$

$$< k|L_2 l > = < L_2 k|l > . \tag{33}$$

The boundary conditions which are imposed to obtain a well defined inner product turn out to be sufficient to make L_1 and L_2 Hermitian.

We will now determine the asymptotic behavior of a vector field $k_A \in V$ which satisfies $L_1 k_A = 0$ or $L_2 k_A = 0$. We start from the fact that, when r is large the components of k_A become:

$$k_r = O(\frac{1}{r^m}), \quad k_\alpha = O(\frac{1}{r^n}), \quad m \geq \frac{p}{2} + 2, \quad n \geq \frac{p}{2}. \tag{34}$$

To find the possible values of m and n, one can expand the components in powers of $1/r$ (which can be done when r is sufficiently large):

$$k_r = \frac{f(x)}{r^m} + \frac{g(x)}{r^{m+1}} + ..., \tag{35}$$

$$k_\alpha = \frac{f_\alpha(x)}{r^n} + \frac{g_\alpha(x)}{r^{n+1}} + ... \tag{36}$$

After plugging the above expansions into the differential equations, one can group the terms according to their powers of $1/r$ and solve equations order by order. The lowest order equations, which are sometimes called the indicial equations, determine the constants m and n. At large r, the components of $\nabla^2 k_A$ can be calculated as (we set the inverse radius $a = 1$ from now on):

$$\nabla^2 k_r = r^2 \partial_r^2 k_r + \frac{1}{r^2}\delta^{\alpha\beta}\partial_\alpha\partial_\beta k_r + k_r + (p+4) r \partial_r k_r - \frac{2}{r^3}\delta^{\alpha\beta}\partial_\alpha k_\beta \\ + O(\frac{1}{r^{p+m+1}}) + O(\frac{1}{r^{p+n+2}}), \tag{37}$$

$$\nabla^2 k_\alpha = r^2 \partial_r^2 k_\alpha + \frac{1}{r^2}\delta^{\gamma\beta}\partial_\gamma\partial_\beta k_\alpha - (p+1) k_\alpha + p r \partial_r k_\alpha + 2r\partial_\alpha k_r \\ + O(\frac{1}{r^{p+n+1}}) + O(\frac{1}{r^{p+m-2}}), \tag{38}$$

where the suppressed terms, which arise due to the deviation of the metric from the AdS soliton background, will not affect the indicial equation. The terms $\frac{1}{r^2}\delta^{\alpha\beta}\partial_\alpha\partial_\beta k_r$ and $\frac{1}{r^2}\delta^{\gamma\beta}\partial_\gamma\partial_\beta k_\alpha$, appeared in (37) and (38), will only contribute to the higher order equations, even though they survive when the deviation from the background is zero. Thus, they can also be ignored in finding the possible values of m and n. The fact that the vector fields are divergence free, $\nabla_A k^A = 0$, gives the following relation among the components:

$$r^2\partial_r k_r + (p+2)rk_r + \frac{1}{r^2}\delta^{\alpha\beta}\partial_\alpha k_\beta + O(\frac{1}{r^{p+m}}) + O(\frac{1}{r^{p+n+1}}) = 0, \tag{39}$$

where the suppressed terms have higher powers of $1/r$ compared to the ones that are written. Now, using (39) for the last term of (37), one can show that the lowest order $1/r$ equations, obtained from the r-components of the differential equations $L_1 k_A = 0$ and $L_2 k_A = 0$, give the following quadratic equations for m:

$$L_1 k_A = 0 \qquad \Rightarrow \qquad m^2 - (p+3)m + (p+4) = 0, \tag{40}$$

$$L_2 k_A = 0 \qquad \Rightarrow \qquad m^2 - (p+3)m + (3p+6) = 0. \tag{41}$$

Fortunately, all the terms which has the constant n turn out to be higher order and we finally end up with equations for m.

For the vector fields satisfying $L_1 k_A = 0$, the constant m will have the following values.

$$m_1 = 1, \qquad m_2 = p+4. \tag{42}$$

The first root is not acceptable since it is not consistent with the boundary conditions imposed on the vector fields. Thus, we set $m = p+4$. Using (38) in the α-component of the equation $L_1 k_A = 0$, one can obtain a quadratic equation for n,

$$n^2 - (p-1)n - (2p+2) = 0, \tag{43}$$

which has the single positive root $n = p+1$. Therefore, the solutions $L_1 k_A = 0$, $k_A \in V$, fall off like:

$$k_r = O(\frac{1}{r^{p+4}}), \qquad k_\alpha = O(\frac{1}{r^{p+1}}). \tag{44}$$

For the vector fields obeying $L_2 k_A = 0$, the roots of m can be read off from (41) as:

$$m_1 = 3, \qquad m_2 = p+2. \tag{45}$$

The first root is not acceptable for the cases $p > 1$, since it is not consistent with the boundary conditions, and the roots become equal for $p = 1$. Therefore, we set

$m = p + 2$. Then, (38) can be used in the α-component of $L_2 k_A = 0$, to obtain the following quadratic equation for n:

$$n^2 - (p-1)n = 0. \tag{46}$$

This has the single positive root $n = p - 1$ and therefore the solutions of $L_2 k_A = 0$, $k_A \in V$, become for large r

$$k_r = O(\frac{1}{r^{p+2}}), \quad k_\alpha = O(\frac{1}{r^{p-1}}). \tag{47}$$

Till now we have fixed the boundary conditions which ensure that the operators L_1 and L_2 are Hermitian and determined the asymptotic behavior of their zero modes. We will now try to obtain further information about the zero modes, since our final aim is to use these operators in solving certain Dirichlet problems which will possibly imply existence of the Killing vectors. It would be nice if our operators L_1 and L_2 turned out to be invertible i.e. they did not have any zero modes. This will be the case for L_1, and for L_2 we will be able to determine all the zero modes.

We proceed by showing that the Hermitian differential operator L_1 does not have any zero modes. When $k_A \in V$, the fact that its divergence free and (24) give

$$\nabla^A \nabla_B k_A = -l^2 k_B, \tag{48}$$

which can be used to rewrite the equation $L_1 k_A = 0$ as:

$$\nabla^A (\nabla_A k_B + \nabla_B k_A) = 0. \tag{49}$$

We contract the above equation with k^B and integrate over all M:

$$\int_M k^B \nabla^A (\nabla_A k_B + \nabla_B k_A) = 0. \tag{50}$$

Integrating by parts, one can easily obtain, for any vector field satisfying $L_1 k_A = 0$ and $\nabla_A k^A = 0$,

$$-\frac{1}{2} \int_M (\nabla^A k^B + \nabla^B k^A)(\nabla_A k_B + \nabla_B k_A) + \int_{\partial M} n^A k^B (\nabla_A k_B + \nabla_B k_A) = 0. \tag{51}$$

Now, the vector fields in V fall off like (44) when they obey $L_1 k_A = 0$. This ensures that the surface integrals vanish which in turn implies,

$$\nabla_A k_B + \nabla_B k_A = 0, \tag{52}$$

since the remaining integrand in (51) is nowhere negative. Therefore, k_A is a Killing vector. By the asymptotics (44), both k_A and $\nabla_A k_B$ vanishes on ∂M. However, it is very well known that if a Killing field and its derivative vanish at any point, then the Killing field vanishes everywhere [17]. Thus $k_A = 0$ and L_1 has no zero modes with the given boundary conditions.

Let us now try to learn about the zero modes of the operator L_2. We will simply repeat the above steps, to obtain a relation like (52). Using (48), the equation $L_2 k_A = 0$ can be rewritten as,

$$\nabla^A(\nabla_A k_B - \nabla_B k_A) = 0. \tag{53}$$

We first contract the above equation with k^B and integrate over all M. Then, integrating by parts, we obtain for any vector field satisfying $L_2 k_A = 0$ and $\nabla_A k^A = 0$:

$$-\frac{1}{2}\int_M (\nabla^A k^B - \nabla^B k^A)(\nabla_A k_B - \nabla_B k_A) + \int_{\partial M} n^A k^B (\nabla_A k_B - \nabla_B k_A) = 0. \tag{54}$$

The surface integrals vanish due to the asymptotic behavior (47). The remaining integrand is positive definite which implies that

$$\nabla_A k_B - \nabla_B k_A = 0. \tag{55}$$

Therefore the zero modes of the Hermitian operator L_2 are closed one-forms. Since $\nabla_A k^A = 0$ they are also co-closed and thus harmonic one-forms.

One can obtain further information about these zero modes by recalling that a closed one-form can be written as a linear combination of an exact one-form and first cohomology classes of the manifold M. The cohomology classes are related only to the topology of the manifold. In our case, we have assumed that, outside a compact region, the topology of M become globally the AdS soliton i.e. $T^p \times S^1$ times a real line (the coordinate r). Therefore, near the asymptotic region, the first cohomology classes are spanned at most [5] by the one-forms dx^α. However, these forms cannot be the zero modes of L_2 since they do not obey the boundary conditions imposed on the vector fields belonging to V. We will show in a moment that the exact forms should also be excluded. Therefore, the zero modes of L_2 are the harmonic representatives of first cohomology classes of the interior compact region.

To see that the exact forms cannot be the zero modes of L_2 we write $k_A = \nabla_A f$, for some function f. By the asymptotic behavior of the zero modes of L_2 determined

[5] In the interior region the topology may change such that some of the one-forms dx^α become also exact. This is the case, for instance, for the one-form $d\tau$ in the solution (8).

in (47), f should behave for large r like,

$$f = O(\frac{1}{r^{p+1}}). \tag{56}$$

Since k_A is also divergence free, f should satisfy the Laplace equation: $\nabla^2 f = 0$. However, any solution of the Laplace equation which obeys (56) should be zero. This can easily be seen by nothing the following relation which is valid for any function obeying Laplace equation:

$$0 = \int_M f\nabla^2 f = -\int_M (\nabla_A f)(\nabla^A f) + \int_{\partial M} n^A f(\nabla_A f). \tag{57}$$

The last surface integral vanishes due to the behavior (56) of f at large r. Since the remaining integrand in the right side is positive definite, one obtains $\nabla_A f = 0$. Therefore, f is a constant and the constant vanishes since f vanishes at $r \to \infty$. This shows that exact forms should be excluded from the list of the zero modes of L_2.

The Laplacian

Having studied properties of the differential operators acting on divergence free vector fields, we finally turn to the space of functions. Let W be the space of square integrable functions endowed with the following inner product,

$$< f|g >= \int_M fg. \tag{58}$$

The inner product is well defined, since the functions in W obey at large r (due to square integrability):

$$f \in W \qquad \Rightarrow \qquad f = O(\frac{1}{r^{p/2+1}}). \tag{59}$$

The differential operator that we will consider on W is the Laplacian. One can easily check that, due to the asymptotic behavior of functions determined above, ∇^2 is an Hermitian operator

$$< f|\nabla^2 g >=< \nabla^2 f|g > . \tag{60}$$

As before, we try to determine the zero modes of ∇^2 and see if it is invertible on W. The asymptotic behavior of the zero modes can be determined by calculating

$$\nabla^2 f = r^2\partial_r^2 f + \frac{1}{r^2}\delta^{\alpha\beta}\partial_\alpha\partial_\beta f - (p+2)r\partial_r f + O(\frac{1}{r^{p+m+1}}), \tag{61}$$

where $f = O(1/r^m)$. The indicial equation which follows from the above formula reads

$$m(m+1) - (p+2)m = 0. \tag{62}$$

The only solution which is consistent with the boundary condition (59) is $m = p + 1$. Therefore, zero modes of ∇^2 in W fall off like $1/r^{p+1}$. However, repeating the above arguments which has been used to show that the exact forms cannot be the zero modes of L_2, one can easily see that the zero modes should actually be zero. Therefore the Laplacian ∇^2 is an invertible operator on W.

4 Existence of harmonic functions and commuting Killing vectors

In the previous section we have considered two vector spaces V and W, endowed with well defined inner products, and studied properties of the differential operators L_1, L_2 and ∇^2. We have shown that all operators are Hermitian and determined the zero modes. The operators L_1 and ∇^2 are shown to have no zero modes and thus are invertible. In this section we will try to use these operators to show the existence of harmonic functions and Killing vectors.

Existence of harmonic functions

Although ∇^2 has no zero modes in W, there may be solutions of $\nabla^2 f = 0$, where $f \notin W$. Consider the following Dirichlet problem

$$\nabla^2 f = 0, \quad f \to b_i x^i \quad as \quad r \to \infty, \tag{63}$$

where b_i are arbitrary constants. Note that the index i runs from 1 to $d-2$. Since the coordinates x^i are periodically identified with the period L, the function f is not well defined for finite L. However, the constant L is a free parameter and one can let $L \to \infty$. Indeed, as discussed in section 2, this limit corresponds to the AdS soliton. On the other hand one cannot include the coordinate τ in (63) since it's period β is fixed (β corresponds to either compactification scale or the inverse temperature of the boundary conformal field theory). Note also that $f \notin W$.

To solve the above Dirichlet problem we first define a trial function t such that near the asymptotic boundary (where the coordinates x^α and r are well defined) it becomes $t = b_i x^i$. We smoothly extend t to be zero through the interior regions (where the coordinates x^α and r may not be well defined) and thus obtain a well defined function on M. We find the solution of (63) by writing $f = l + t$, and solving for l obeying

$$\nabla^2 l = -\nabla^2 t, \quad l \to 0 \quad as \quad r \to \infty. \tag{64}$$

Using (61), one can check that $\nabla^2 t = O(1/r^{p+2}) \in W$ [6] and thus there is a unique solution for $l \in W$, since ∇^2 is invertible. By (59), $l = O(1/r^{p/2+1})$. To obtain more accurate information about the asymptotic behavior of l, we expand it at large r as

$$l = \frac{f(x)}{r^m} + \frac{g(x)}{r^{m+1}} + ... \qquad m \geq \frac{p}{2} + 1. \tag{65}$$

Now, (61) implies that the lowest order terms of $\nabla^2 l$ fall off like $1/r^m$. Since l obeys (64), these terms should either cancel each other or be equal to the same order terms coming from the right hand side. Thus, either m should obey the indicial equation (62) and $m = p+1$ or $m = p+2$. This implies, at least, $l = O(1/r^{p+1})$ which, furthermore, gives the following solution to the Dirichlet problem (63):

$$f = b_i x^i + O(\frac{1}{r^{p+1}}). \tag{66}$$

One can also try to solve for l by constructing a Green's function, $G(r,r')$, for ∇^2. The asymptotic behavior of $G(r,r')$ can be deduced from the asymptotics of the possible zero modes since it obeys the Laplace equation except at some singular points. Inverting (64) as $l(r) = -\int G(r,r')\nabla^2 t(r')$ and letting $r \to \infty$, one finds that $l = O(1/r^{p+1})$ and thus f has the same asymptotic behavior (66). Since f obeys $\nabla^2 f = 0$, it is an harmonic function $d * df = 0$. Note also that the one-form df is also harmonic $(d\delta + \delta d)df = 0$, where δ is the adjoint of d. On the other hand, since in the construction we have introduced arbitrary constants b_i, there are actually $d-2$ independent harmonic functions. We will denote the harmonic function obtained by the choice $b = (0,0,..1,0..0)$ by $f_{(i)}$, where in b only the i'th entry is one. All these are valid for *all* instanton solutions.

Existence of Killing vectors

The Hermitian operator L_1 is invertible on V. However, there may be solutions of $L_1 k_A = 0$, where k_A is a divergence free vector field which does not belong to V. Let us try to see if the following Dirichlet problem has a solution,

$$L_1 k_A = 0, \qquad \nabla_A k^A = 0$$

$$k^\alpha \to a^\alpha, \qquad k^r \to 0, \qquad as \qquad r \to \infty, \tag{67}$$

where a^α are arbitrary constants. Note that, the *contravariant* components of the vector field are fixed on the boundary. On the other hand, the covariant components

[6] One may naively claim from (61) (by setting $m = 0$) that $\nabla^2 t = O(1/r^{p+1})$. However, a carefull analysis for t shows that $O(1/r^{p+1})$ terms cancel each other in (61) which implies $\nabla^2 t = O(1/r^{p+2})$.

become, for example, $k_\alpha = O(r^2)$ which shows that $k_A \notin V$.

To solve this problem, we first define a trial vector field t_A with the following properties. Near the asymptotic boundary, the components $t_\alpha(x,r)$ are smooth functions of coordinates x^α, r such that they vanish when $r < R_-$ and become exactly $a^\alpha r^2$ when $r > R_+$, where R_- and R_+ are sufficiently large numbers with $R_- < R_+$. The component function, $t_r(x,r)$, is defined to be zero when $r < R_-$ and will be chosen to give $\nabla_A t^A = 0$ when $r > R_-$. We extend t_A to be zero trough the interior regions, where the coordinates x^α and r may not be well defined. This gives us a well defined divergence free vector field over all M.

We introduce a new vector field l_A by writing the desired solution of (67) as:

$$k_A = l_A + t_A. \tag{68}$$

By the construction of the trial vector field t_A, one can solve the Dirichlet problem (67) by working the solutions l_A of,

$$L_1 l_A = -L_1 t_A, \quad l_A \to 0, \quad as \quad r \to \infty. \tag{69}$$

We will show that $L_1 t_A \in V$ when $p > 3$. Since L_1 is an invertible operator on V, there is a unique solution of (69) with $l_A \in V$. This will give a unique solution of the Dirichlet problem (67) for $p > 3$.

The α-component t_α of the trial vector field is fixed by construction. Let us determine the behavior of t_r at large r. As mentioned above, t_r is chosen to obtain a divergence free vector field, thus satisfies a first order differential equation. Using (39) for the components of t_A, we obtain at large r

$$t_r = O(\frac{1}{r^p}). \tag{70}$$

Using (37) and (38), one can see that, in $L_1 t_A$, the terms which come from the background and contain the constants a^α cancel each other. The remaining terms can be determined to fall off like

$$(L_1 t)_r = O(\frac{1}{r^p}), \quad (L_1 t)_\alpha = O(\frac{1}{r^{p-1}}). \tag{71}$$

This shows that $L_1 t_A \in V$ when $p > 3$, since it obeys the boundary conditions (29) only on that range. As mentioned above, L_1 can be inverted to obtain a unique solution $l_A \in V$. Therefore, the components of l_A obey at least (29). However, we will need more accurate information about the asymptotic behavior of l_A. To obtain this information, we start from the fact that at large r:

$$l_r = O(\frac{1}{r^m}), \quad l_\alpha = O(\frac{1}{r^n}), \quad m \geq \frac{p}{2} + 2, \quad n \geq \frac{p}{2}, \tag{72}$$

and expand the components like in (35). The lowest order terms of the r-component of $L_1 l_A$ can be seen to fall off like $1/r^m$. At this stage there are two possibilities; these terms may cancel each other or they may be equal to the same order terms coming from the right hand side of (69) i.e. $-(L_1 t)_r$. For these terms to cancel each other, m should satisfy exactly the same indicial equation (40) obtained for the zero modes. If this is the case then $m = p + 4$, but now in $(L_1 l)_r$ there is no term which can cancel the lowest order terms coming from $(L_1 t)_r$. Thus the second possibility should be true. The asymptotic behavior of $(L_1 t)_r$ implies that $m = p$ and $l_r = O(1/r^p)$. The same arguments can be repeated for the α-components to obtain:

$$l_r = O(\frac{1}{r^p}), \quad l_\alpha = O(\frac{1}{r^{p-1}}). \tag{73}$$

This shows that there is a unique solution of $L_1 k_A = 0$, $\nabla_A k^A = 0$, given by $k_A = l_A + t_A$, which asymptotically becomes (contravariant components):

$$k^r = O(\frac{1}{r^{p-2}}), \quad k^\alpha = a^\alpha + O(\frac{1}{r^{p+1}}). \tag{74}$$

Note that, to be able to solve the Dirichlet problem, we are forced to impose $p > 3$ i.e. the dimension of M should be greater than 5.

We now try to establish that (74) is a Killing vector. To show this, one can use the equation (51), which is valid for any vector field obeying $L_1 k_A = 0$ and $\nabla_A k^A = 0$. If the surface integral in (51) vanishes, then one has $\nabla_A k_B + \nabla_B k_A = 0$, since the remaining terms in (51) are positive definite. There is *only one* contribution to the surface integral, coming from $n^r k^\alpha (\nabla_\alpha k_r + \nabla_r k_\alpha)$, which does not vanish. Since the volume form of ∂M grows like r^{p+1}, $n^r = O(r)$ and $k^\alpha = O(1)$, the surface integral integral of this term vanishes when

$$\nabla_\alpha k_r + \nabla_r k_\alpha = O(\frac{1}{r^{p+3}}). \tag{75}$$

However, one can easily check that this condition is *not* satisfied by the solution (74) of the Dirichlet problem. The solution (74) obeys, instead, $\nabla_\alpha k_r + \nabla_r k_\alpha = O(1/r^p)$, which shows that the surface integral may not vanish in general. Note that $a^\alpha \frac{\partial}{\partial x^\alpha}$

is already an asymptotic Killing vector field of any possible instanton like solution. To see on which spaces the surface integral vanishes, let us concentrate on a class of instanton solutions obeying more restrictive boundary conditions such as

$$h_{\alpha\beta} = O(\frac{1}{r^{p+2}}), \quad h_{\alpha r} = O(\frac{1}{r^{p+3}}), \quad h_{rr} = O(\frac{1}{r^{p+6}}). \tag{76}$$

Compared to the asymptotic behavior (13), the metrics of this family fall off three power of r faster. In order to take into account of this difference in the previous formulas, one should simply shift p by 3. This implies that the Dirichlet problem (67) can now be solved in *any* dimensions and the solution (74) becomes asymptotically

$$k^r = O(\frac{1}{r^{p+1}}), \quad k^\alpha = a^\alpha + O(\frac{1}{r^{p+4}}). \tag{77}$$

By plugging (77) into the identity (51), one can see that the surface integrals do vanish. Therefore, (77) is indeed a Killing vector. On the othar hand, since the constants a^α are completely arbitrary, there are actually $p+1$ independent Killing vectors. We will denote the Killing vector field which is obtained by the choice $a^\alpha = (0,0..1,..0)$ by $k^{(s)A}$ where $p+1 \geq s \geq 1$ and in a^α only the s'th entry is one. We remind the reader that the proof only applies to the family (76).

By defining

$$z_A \equiv [k^{(s)}, k^{(s')}]_A = k^{(s)B} \nabla_B k^{(s')}_A - k^{(s')B} \nabla_B k^{(s)}_A, \tag{78}$$

one can easily show that the Killing vectors of this family also commute. The asymptotic behavior (77) implies that

$$z_r = O(\frac{1}{r^{p+3}}), \quad z_\alpha = O(\frac{1}{r^{p+2}}). \tag{79}$$

Since the commutator of two Killing vectors is again a Killing vector, z_A is a Killing vector. Being a Killing vector and having the above asymptotic behavior, $z_A \in V$ and $L_1 z_A = 0$. However, since L_1 is invertible, $z_A = 0$. Thus the Killing vectors commute with each other.

Operator L_2

We have studied Dirichlet problems related to the operators L_1 and ∇^2. As we have discussed, the existence of the solutions imply the existence of certain geometric properties. There is another operator, L_2, which we have studied in section 2 and not considered yet in the context of Dirichlet problems. Since the Hermitian differential operator L_2 is not invertible on V, one cannot naively solve a Dirichlet problem using L_2 (at least following the steps discussed so far). It seems that one should also make some topological assumptions to obtain an invertible operator. Here, we simply note that the one-forms $\nabla_A f_{(i)}$ are divergence free and they are also eigenvectors of L_2 with zero eigenvalue, $L_2 \nabla_A f_{(i)} = 0$. Note also that $\nabla_A f_{(i)} \notin V$.

Relation between $k_A^{(i)}$ and $df_{(i)}$

We conclude this section by showing that all but one of the Killing vectors are duals of harmonic one-forms $df_{(i)}$:

$$k^{(i)A} \nabla_A f_{(i')} = \delta^i_{i'}. \tag{80}$$

Note that at $r = \infty$ the Killing vector $k^{(i)A}$ and the function $f_{(i)}$ equal to $\frac{\partial}{\partial x^i}$ and x^i, respectively. To prove (80), using $\nabla_{(A} k^{(i)}_{B)} = 0$, $L_1 k_{(i)A} = 0$ and $L_2 \nabla_A f_{(i)} = 0$, we calculate

$$\begin{aligned}\nabla^2 \left(k^{(i)A} \nabla_A f_{(i')} \right) = \left(\nabla^2 k^{(i)A} \right) \nabla_A f_{(i')} + k^{(i)A} \left(\nabla^2 \nabla_A f_{(i')} \right) + \\ 2 \left(\nabla^B k^{(i)A} \right) \left(\nabla_B \nabla_A f_{(i')} \right) = 0,\end{aligned} \tag{81}$$

which shows that the functions $k^{(i)A} \nabla_A f_{(i')}$ obey the Laplace equation. One can now use (57) which is valid for any function obeying Laplace equation. We focus on the surface integrals. If one writes $g = k^{(i)A} \nabla_A f_{(i')}$, then using the asymptotic behavior (77) and (66) of $k^{(i)A}$ and $f_{(i)}$, respectively, one can obtain

$$g = \delta^i_{i'} + O(\frac{1}{r^{p+1}}). \tag{82}$$

Then, the surface integrals in (57), obtained after plugging g into, can be written explicitly as:

$$\delta^i_{i'} \int_{\partial M} n^A \nabla_A g + \int_{\partial M} O(\frac{1}{r^{p+1}}) n^A \nabla_A g. \tag{83}$$

The first term in the above integrals vanishes since it can be written as the integral of $\nabla^2 g$ over all M which is zero, and the second integral vanishes since the volume form on ∂M grows like r^{p+1}, $n^A = O(r)$, $\nabla_A g = O(1/r^{p+1})$ which imply that the integrand vanishes as $r \to \infty$. Therefore, the surface integrals vanish in (57). The remaining integrand in (57) is positive definite which implies that $\nabla_A g = 0$. Therefore, g is constant and the constant is $\delta^i_{i'}$ since g reaches this value at $r \to \infty$. This proves (80), which in turn shows that the Killing vectors $k_A^{(i)}$ and the one-forms $df_{(i)}$ cannot *vanish* at any point. Thus, all but one of the Killing vectors do not have any fixed points and act on M freely. This fact is important since it enables one to choose global coordinates adapted to Killing vectors, which are the functions $f_{(i)}$ in our case.

5 Conclusions

In this paper we have tried to obtain certain geometric properties of AdS instantons. After determining the asymptotic behavior of the metric, we have studied certain Hermitian differential operators, defined on suitable Hilbert spaces. Using these operators, we have shown that on a d-dimensional instanton there are $d-2$ harmonic functions. For a class of instatons, we have also proved the existence of $d-1$ commuting Killing vectors. Furhermore, all but one of the Killing vectors are shown to have no fixed points.

These results may be usuful in proving the semiclassical stability of AdS soliton. We believe that the existence proof of the Killing vectors on the family (76) can be generalized to all instanton solutions by choosing more appropriate trial vector fields in solving the Dirichlet problem corresponding to L_1. If this could be done, one can introduce globally well defined coordinates adapted to the Killing vectors which act on the manifold freely, and try to solve the Einstein equations in these coordinates as an attempt to obtain all possible instanton solutions.

The existence of an instanton solution does not necessarily imply that the AdS soliton is semiclassically unstable. This solution may well describe the decay of another unstable state to AdS soliton. To see that a ground state is semiclassically unstable, one should study the quadratic fluctuations around an instanton solution. The negative action modes which may arise in this quadratic approximation is a sign of semiclassical instability.

In the the proofs of semiclassical stability of Minkowski space and the positive energy theorem for asymptotically flat spaces given in [4], the Hermitian operators and their Green's functions played the crucial role. The operators introduced in this paper may also be useful in studying the positive energy conjecture of Horowitz and Myers. In the present paper, the metric is positive definite and this plays the key role at each step in deriving the results. In the Lorentzian context, one can choose an Euclidean initial value hypersurface and try to adapt the arguments presented here to obtain information about this Euclidean hypersurface.

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